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Geometric Phase in Quaternionic Quantum Mechanics

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Quaternion quantum mechanics is examined at the level of unbroken $SU(2)$ gauge symmetry. A general quaternionic phase expression is derived from formal properties of the quaternion algebra.

1. QUATERNION QUANTUM MECHANICS

Quantum mechanics defined over general algebras have been conjectured since 1934 (Jordan *et al.*, 1934). The use of quaternions was suggested in a proper manner by Birkoff and von Neumann in 1936, when they noted that the propositional calculus implies in a representation of pure states of a quantum system by rays on a Hilbert space defined over any associative division algebra (Birkoff and von Neumann, 1936). This means that quantum theory would be limited to the real, complex and quaternion algebras. However, the spinor representations of the rotation group requires the existence of solutions of quadratic algebraic equations related to the invariant operators, which are guaranteed only over a complex algebra (Chevalley, 1955). The development of quaternion quantum mechanics started with D. Finkelstein in 1959, its relativistic and particle aspects were studied by G. Emch and E. J. Schremp (Emch, 1963; Finkelstein, 1959; Finkelstein *et al.*, 1962, 1963; Schremp, 1967). A comprehensive reference list can be found in Alder (1995).

In an attempt to interpret quaternion quantum mechanics, C. N. Yang suggested that the isospinor symmetry should be contained, in the group of automorphisms of the quaternion algebra (Finkelstein, 1959; Finkelstein *et al.*, 1962, 1963). This interpretation may be seen through the following argument: Suppose

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that the spin angular momentum $\vec{M}$ associated with the spinor representations of the $SO(3)$ subgroup of the Lorentz group, and the isospin angular momentum $\vec{I}$ given by a representation of the group $SU(2)$, are both present in a single state. These groups are isomorphic and their spinor representations are given by the Pauli matrices acting separately on the spinor space $\mathcal{M}$ and the isospinor space $\mathcal{I}$ respectively, generated by two complex spinor basis $(1, i)$ and $(1, j)$. The direct sum $\mathcal{M} \oplus \mathcal{I}$ does not close as an algebra, except if the product $k = ij$ is introduced producing a quaternion algebra, whose automorphisms becomes the carrier of the combined spin–isospin symmetry.

According to this interpretation, quaternion quantum mechanics would be effective at the energy level in which spin and isospin symmetries remain combined. In addition, when this combined symmetry breaks down at lower energies, we expect to recover two spin degrees of freedom (Emch and Jadczyk, 1998; Hazenfrantz and 't Hooft, 1976; Jackiw and Rebbi, 1976; Singleton, 1995; Vachaspati, 1997).

The existence and effectiveness of quaternion quantum mechanics at higher energies must be experimentally verified. In one of the experiments proposed by A. Peres, a neutron interferometer with thin plates made of materials with varying proportions of neutrons and protons is used, where the phase difference in one or another case is measured (Peres, 1979). This experiment can be adapted to a variable beam intensity, so that in principle the phase difference of the complex and the quaternion theories at different levels of energy could be detected.

Since quaternions keep a one-to-one correspondence with space–time vectors, the quaternion phase can also be set in a one-to-one correspondence with a rotation subgroup of the Lorentz group. In this sense, the geometric quaternion phase is truly geometrical as compared with the geometric complex phase, which is defined on a projective space (Berry, 1988). This space–time interpretation means that the integration of the quaternionic quantum phase, along a closed loop in space–time, can be associated with the space–time curvature, suggesting a quantum gravitational effect.

Taking the quaternion wave function Ψ as a solution of Schrödinger's equation defined with an anti-Hermitian quaternionic Hamiltonian operator H, then the quaternionic dynamical phase ω can be described as

$$\oint_C \omega^{-1}\, d\omega = -\int_C \langle \Psi | H | \Psi \rangle \, dt \tag{1}$$

where $\langle , \rangle$ denotes the quaternionic Hilbert product. Adler and Anandan (1996) have proposed a solution of this integral as given by

$$\omega(t) = T \ \exp\left(-\int_0^t \langle \Psi | H | \Psi \rangle \, dv \right), \tag{2}$$

where T denotes a constant quaternion representing a time ordering factor.

On the other hand, the geometric phase $\tilde{\omega}$ is determined by

$$\oint_C \tilde{\omega}^{-1}\, d\tilde{\omega} = -\int_C \left\langle \Psi \,\middle|\, \frac{d\Psi}{dt} \right\rangle dt, \tag{3}$$

Again, according to Adler and Anandan (1996) this may be integrated to give the following result

$$\tilde{\omega}(t) = T\ \exp\left(-\int_0^t \left\langle \Psi \,\middle|\, \frac{d\Psi}{dv} \right\rangle dv \right). \tag{4}$$

The question we address ourselves concerns with the generality of this solution as represented by a Volterra integral combined with a fixed ordering factor T. We shall see that the left-hand side of (3) has a more general solution expressed by the imaginary quaternionic exponential function.

We start by examining the meaning and uniqueness of the quaternionic line integral (3). Denoting a quaternion function of a quaternion variable $X = X_\alpha\, e^\alpha$ in a quaternion basis e^α by $f(X) = U_\alpha\, e^\alpha$ where U_α are real components, we may define the left and right line integrals respectively by[4]

$$\int_C f(X)\, dX = e^\alpha\, e^\beta \int_C U_\alpha\, dx_\beta, \qquad \int_C dX\ f(X) = e^\beta\, e^\alpha \int_C U_\alpha\, dx_\beta.$$

These integrals are not necessarily equal:

$$\int_C f(X)\, dX - \int_C dX\ f(X) = \sum \epsilon_{ijk} e^k \int_C (U_i\, dx_j - U_j\, dx_i).$$

However, for a closed loop C, it is readily seen that this difference vanishes as a consequence of Green's theorem in the plane $(i,\ j)$. Therefore, the phase expression in the left-hand side of (3) is uniquely defined.

To obtain a solution of (3) that is more general than (4) we need to understand why it is important to be a division algebra.

2. HARMONIC FUNCTIONS

The division algebra condition $|AB| = |A||B|$ is a basic requirement to perform the limit operations of products leading to the standard real and complex mathematical analysis. In the complex case, the limit operation is independent of the direction, leading to the Cauchy–Riemann equations. However, when we attempt to extend the same definitions to the quaternion algebra, the generalized

[4] Greek indices run from 0 to 3 and small Latin indices from 1 to 3. The quaternion multiplication table is $e^i\, e^j = -\delta^{ij} + \sum \epsilon^{ijk}\, e^k$, $e^i\, e^0 = e^0\, e^i = e^i$ and $e^0\, e^0 = e^0 = 1$. Quaternion conjugate is denoted with overbar: $\overline{e^i} = -e^i$, $\overline{e^0} = e^0$. The quaternion norm is $|X|^2 = X\bar{X}$. The sum convention applies throughout.

Cauchy–Riemann conditions become so restrictive that only a few trivial functions survive (see the appendix for a brief review) (Fueter, 1932, 1936, 1937; Evans *et al.*, 1992; Ferraro, 1938; Ketchum, 1928; Khaled Abdel-Khalek, 1996; Nash and Joshi, 1987). The less restrictive harmonicity condition is (Evans *et al.*, 1992)

$$\sum \delta^{ij} \frac{\partial^2 U_\alpha}{\partial X_i \, \partial X_j} + \frac{\partial^2 U_\alpha}{\partial X_0^2} = \Box^2 U_\alpha = 0, \tag{5}$$

Quaternion harmonicity can be easily implemented by the introduction of the quaternionic slash differential operator $\not\partial = \sum e^\alpha \; \partial_\alpha = \sum e^\alpha \; \partial/\partial X_\alpha$, such that $\Box^2 = \not\partial \, \overline{\not\partial}$. This operator may act on the right and on the left of a quaternion function $f(X)$, giving

$$\not\partial f(X) = \left(\frac{\partial U_0}{\partial X_0} + \sum \frac{\partial U_i}{\partial X_0} \, e^i \right) + \sum \left[\frac{\partial U_0}{\partial X_i} \, e^i - \sum \frac{\partial U_i}{\partial X_j} (\delta^{ij} - \epsilon^{ijk} \, e^k) \right],$$

$$f(x) \not\partial = \left(\frac{\partial U_0}{\partial X_0} + \sum \frac{\partial U_i}{\partial X_0} \, e^i \right) + \sum \left[\frac{\partial U_0}{\partial X_i} \, e_i - \sum \frac{\partial U_i}{\partial X_j} (\delta^{ij} + \epsilon^{ijk} \, e^k) \right]$$

It is clear that $\not\partial f(X) \neq f(X) \not\partial$, unless the condition

$$\frac{\partial U_i}{\partial X_j} = \frac{\partial U_j}{\partial X_i}, \tag{6}$$

holds. Three classes of harmonic functions may be defined:

a) The left harmonic functions, characterized by $\not\partial f(X) = 0$

$$\frac{\partial U_0}{\partial X_0} = \sum_i \frac{\partial U_i}{\partial X_i},$$

$$\frac{\partial U_k}{\partial X_0} + \frac{\partial U_0}{\partial X_k} = -\sum_{ij} \epsilon^{ijk} \, \frac{\partial U_i}{\partial X_j}.$$

b) The right harmonic functions, such that $f(X) \not\partial = 0$

$$\frac{\partial U_0}{\partial X_0} = \sum_i \frac{\partial U_i}{\partial X_i},$$

$$\frac{\partial U_k}{\partial X_0} + \frac{\partial U_0}{\partial X_k} = \sum_{ij} \epsilon^{ijk} \, \frac{\partial U_i}{\partial X_j}.$$

c) The totally harmonic functions (or simply H functions), characterized by $\not\partial f(X) = 0$ and $f(X) \not\partial = 0$

$$\frac{\partial U_0}{\partial X_0} = \sum_i \frac{\partial U_i}{\partial X_i},$$

$$\frac{\partial U_i}{\partial X_0} = -\frac{\partial U_0}{\partial X_i} \tag{7}$$

$$\frac{\partial U_i}{\partial X_j} = \frac{\partial U_j}{\partial X_i}.$$

The functions belonging to these three classes satisfy the harmonic condition (5).

A nontrivial example of H function is given by an instanton expressed in terms of quaternions (Atiah, 1979). The connection of an anti–self-dual $SU(2)$ gauge field is given by the form

$$\Gamma = \sum_\alpha A_{\alpha(X)}\, dx^\alpha, \tag{8}$$

where $A_0 = \sum U_k\, e^k$, $A_k = U_0\, e^k - \sum \epsilon_{ijk} U_i\, e^j$ and

$$U_0 = \frac{\frac{1}{2} X_0}{1 + |X|^2}, \qquad U_i = \frac{-\frac{1}{2} X_i}{1 + |X|^2},$$

are the components of the quaternion function $f(X) = U_\alpha\, e^\alpha$. We can see that $f(X)$ satisfy the conditions (7) in the region of space–time defined by $\sum X_i^2 = -2X_0$.

The above example is a particular case of a wider class of functions with components

$$U_\alpha = g_\alpha(X)/(1 + |X|^2),$$

where $g_\alpha(X)$ are real functions.

Although (7) could be taken as the definition of quaternion analyticity, there are some functions that are clearly analytic, such as a constant quaternion, which does not satisfy those equations. Therefore, as it happens in the cases of real and complex functions, an analytic quaternion function should be more generally represented by a convergent positive power series, rather than by Eq. (7).

3. POWER SERIES

Given a quaternion function $f(X)$ defined on an orientable three-dimensional hypersurface S with unit normal vector η, we may define two hypersurface integrals.

$$\int_S f(X)\, dS_\eta \quad \text{and} \quad \int_S dS_\eta\, f(X),$$

where $dS_\eta = \sum dS_i\, e^i$ denotes the quaternion hypersurface element with components

$$dS_0 = dX_1\, dX_2\, dX_3, \qquad dS_1 = dX_0\, dX_2\, dX_3,$$

$$dS_2 = dX_0\, dX_1\, dX_3, \qquad dS_3 = dX_0\, dX_1\, dX_2.$$

On the other hand, denoting by $dv = dX_0\, dX_1\, dX_2\, dX_3$ the four-dimensional volume element in a region Ω bounded by S, after integrating in one of the variables, we obtain

$$\int_\Omega \not\partial f(X)\, dv = \int_\Omega e^\alpha\, \partial_\alpha\, e^\beta U_\beta\, dv$$
$$= \int_\Omega \left[\left(\partial_0 U_0 - \sum_i \partial_i U_i \right) + \sum_i (\partial_0 U_i + \partial_i U_0)\, e^i + \epsilon^{ijk}\, \partial_i U_j\, e^k \right] dv.$$

Noting that

$$\int_\Omega \partial_0 U_0\, dv = \int_S U_0\, dS_0, \qquad \int_\Omega \partial_0 U_i\, dv = \int_S U_i\, dS_0,$$
$$\int_\Omega \partial_i U_0\, dv = \int_S U_0\, dS_i, \qquad \int_\Omega \partial_i U_j\, dv = \int_S U_j\, dS_i,$$

it follows that

$$\int_\Omega \not\partial\, f(X)\, dv = \int_S \left[\left(U_0\, dS_0 - \sum \delta^{ij} U_i\, dS_j \right) e^0 + \sum (U_i\, dS_0 + U_0\, dS_i)\, e^i - \sum \epsilon^{ijk} U_i\, dS_j\, e^k \right].$$

A straightforward calculation, shows that this is exactly the same expression of the surface integral

$$\int_S dS_\eta\; f(X) = \sum \int_S U_\alpha\, dS_\beta\, e^\beta\, e^\alpha$$

Therefore, we obtain the result

$$\int_\Omega \not\partial\, f(X)\, dv = \int_S dS_\eta\; f(X) \tag{9}$$

Similary, for the left surface integral we have

$$\int_\Omega f(X) \not\partial dv = \int_S f(X)\, dS_\eta. \tag{10}$$

These integrals are defined for any quaternion functions whose components are integrable and their difference is

$$\sum \epsilon^{ijk}\, e^k \int_S (U_i\, dS_j - U_j\, dS_i) = -\sum \epsilon^{ijk}\, e^k \int_\Omega \left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right) dv,$$

The right-hand side is zero so that only one type of surface integral need to be considered.

The following theorem extends the first Cauchy's Theorem to quaternion functions:

If $f(X)$ satisfy (7) in the interior of a region Ω bounded by a hypersurface S then

$$\int_S f(X)\, dS_\eta = \int_S dS_\eta\; f(X) = 0. \tag{11}$$

This property follows immediately from Eqs. (9) and (10), and the conditions (7). The second Cauchy's theorem is also true for H functions:

If $f(X)$ satisfy the conditions (7) in a region bounded by a simple closed three-dimensional hupersurface S, then for a point P in S, we have

$$f(P) = \frac{1}{\pi^2} \int_S f(X)(X-P)^{-3}\, dS_\eta. \tag{12}$$

The proof is similar to the complex case: The integrand does not satisfy the conditions (7) in Ω as it is not defined at P and consequently the previous theorem does not apply. However the point P may be isolated by a sphere with surface S_0 with center at P and radius ϵ such that it remains inside Ω. Applying (11) to the region bounded by S and S_0 we obtain

$$\int_S f(X)(X-P)^{-3}\, dS_\eta + \int_{S_0} f(X)(X-P)^{-3}\, dS_\eta = 0.$$

Now, the components U_α may be assumed to be differentiable and regular so that we may calculate their Taylor expansions around P:

$$U_\alpha(X) = U_\alpha(P) + \epsilon^\beta \frac{\partial U_\alpha}{\partial x^\beta}\bigg]_P + \cdots.$$

Replacing this in the integral over S_0 and taking the limit $\epsilon \to 0$, it follows that

$$f(P) = \left(\int_S f(X)(X-P)^{-3}\, dS_\eta\right)\left(\int_{S_0} (X-P)^{-3}\, dS_\eta\right)^{-1}. \tag{13}$$

In order to calculate the integral over the sphere S_0 it is convenient to use four-dimensional spherical coordinates $(r, \theta, \phi, \gamma)$, such that $X_0 = r \sin\gamma$, $X_1 = r \cos\gamma \sin\theta \cos\phi$, $X_2 = r \cos\gamma \sin\theta \sin\phi$, and $X_3 = r \cos\gamma \cos\theta$ where $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$, $\gamma \in (-\pi/2, \pi/2)$. The volume element in spherical coordinates is $dv = J\ dr\ d\theta\ d\phi\ d\gamma$ where $J = -r^3 \cos^2\gamma \sin\theta$ is the Jacobian determinant.

The unit normal to the spherical hypersurface centerd at P and with radius ϵ can be written as $\eta = (X - P)/\epsilon$ so that $(X - P)^{-3} = \overline{\eta}^3/\epsilon^3$ and

$$-\int_{S_0} (X - P)^{-3}\, dS_\eta = \int_{S_0} \overline{\eta}^2 \cos^2\gamma \sin\theta \, d\theta \, d\phi \, d\gamma = \pi^2.$$

After replacing in (13), we obtain the proposed result (12). Notice that the power (-3) in (12) is not accidental as it is the right power required to cancel the Jacobian determinant when $\epsilon \to 0$. Now we may prove the following general result:

Let $f(X)$ *be such that it satisfies* (7) *inside a region* Ω *bounded by a surface* S. *Then for all* X *inside* Ω *there exists coefficients* a_n *such that*

$$f(X) = \sum_0^\infty a_n (X - Q)^n. \tag{14}$$

Again, the proof is a straightforward adaptation from the similar complex theorem: If S_0 is the largest sphere in Ω centered at Q, the integral (12) for a point $P = X$ inside Ω gives

$$f(X) = \frac{1}{\pi^2} \int_s f(X')(X' - Q)^{-3}[1 - (X' - Q)^{-1}(X - Q)]^{-3}\, dS'_\eta.$$

It is a simple matter to see that the particular function $f(X) = (1 - X)^{-3}$, with $|X| < 1$ can be expanded as

$$(1 - X)^{-3} = \sum_1^\infty \frac{n(n+1)}{2} X^{n-1} = \sum_{m=0}^\infty \frac{(m+1)(m+2)}{2} X^m. \tag{15}$$

Assuming that $|X - Q| < |X' - Q|$ and using (15), the previously mentioned integrand is equivalent to

$$[1 - (X' - Q)^{-1}(X - Q)]^{-3} = \sum_0^{m=\infty} \frac{(m+1)(m+2)}{2} (X' - Q)^{-m}(X - Q)^m,$$

so that

$$f(X) = \frac{1}{\pi^2} \sum_{m=0}^\infty \frac{(m+1)(m+2)}{2} \int_{S_0} f(X')(X' - Q)^{-3-m}(X - Q)^m \, dS'_\eta.$$

Since η and $(X - Q)$ are proportional, the expression given previously may be written as

$$f(X) = \frac{1}{\pi^2} \sum_{m=0}^\infty \frac{(m+1)(m+2)}{2} \int_{S_0} f(X')(X' - Q)^{-3-m} \, dS'_\eta (X - Q)^m,$$

Defining the coefficients

$$a_m = \frac{1}{\pi^2} \frac{(m+1)(m+2)}{2} \int_{S_0} f(X')(X'-Q)^{-3-m}\, dS'_\eta, \tag{16}$$

we obtain expression (14), showing that all functions satisfying (7) can also be expressed as a convergent positive power series. The converse is not generally true.

4. BACK TO PHASE

Now we may define a quaternion exponential function in terms of convergent power series and in particular the pure imaginary quaternionic exponential to represent the phase. Let us express the solution of (3) as the quaternionic ordered exponential function defined by

$$\tilde{\omega} = P \exp\left(\int_C \tilde{\omega}^{-1}\, d\tilde{\omega} \right)$$

To write this function, consider the quaternion $X = X_0\, e^0 + \sum X_i\, e^i$. With the last three components we may associate the three-vector $\vec{\xi}$, and a pure imaginary quaternion ξ such that $|\xi|^2 = \sum X_i^2 = \vec{\xi} \cdot \vec{\xi}$, where the dot means the Euclidean scalar product. This vector is determined by three angles, which define a unit vector $\vec{\Upsilon} = \vec{\xi}/\sqrt{\vec{\xi} \cdot \vec{\xi}}$ corresponding to the pure imaginary quaternion $\Upsilon = \xi/|\xi|$, such that $\Upsilon^2 = -1$.

With this, we draw the Gauss plane with e^0 in the real axis and Υ in a direction orthogonal to e^0. Then a quaternion X with modulus $|X|$, making an angle γ with e^0 may be expressed as

$$X = X_0\, e^0 + X_i\, e^i = |X|(e^0 \cos\gamma + \Upsilon \sin\gamma).$$

After replacing $\sin\gamma$ and $\cos\gamma$ by the respective power series expansions and after rearranging the terms, we define the quaternion exponential $\mathrm{P}\exp(\Upsilon\gamma)$ by the series within the parenthesis, so that

$$P \exp(\Upsilon\gamma) = \frac{X}{|X|}$$

Therefore, the most general integral of (3) may be expressed as the geometric quaternion phase

$$\tilde{\omega} = P \exp(\Upsilon\gamma) = \cos\gamma + \Upsilon \sin\gamma. \tag{17}$$

Notice that in contrast to (4) there is no fixed direction T but rather the unit direction Υ that varies with the quantum states. Contrary to the complex phase, $\tilde{\omega}$ acts over the quaternion wave functions as an inner automorphism

$$\Psi' = P\exp(\Upsilon\gamma)^{-1}\Psi \ P\exp(\Upsilon\gamma) = \tilde{\omega}^{-1}\Psi\tilde{\omega},$$

Consequently, the quaternion phase is in fact distinct from the complex phase both from the analytic point of view as well as from its geometric interpretation. In the particular case where the vector $\vec{\Upsilon}$ is fixed we obtain a solution equivalent to (4).

The overall conclusion is that in agreement with previous suggestions, quaternion quantum mechanics should be effective at the level of an unbroken $SU(2)$ gauge symmetry. The quaternionic spinor operator transforms under the automorphism of the quaternion algebra, producing a distinct behavior on the phase of the wave functions, as compared with the complex theory.

The neutron interferometry, experiment proposed by Peres (1979) can be modified to accommodate the high-energy interpretation. Accordingly, we suggest a variable beam experiment over plates made of the same material. A higher energy beam should show a qualitative difference from the lower energy case, evidencing the distinction between the quaternion and complex phases.

APPENDIX: BASIC QUATERNIONIC ANALYSIS

Taking a generic quaternion function $f(X) = U_\alpha(X)\, e^\alpha$, and denoting $\Delta f = [f(X+\Delta X) - f(X)]$, the left and right derivatives of $f(X)$ are defined respectively by

$$f'(X) = \lim_{\Delta X\to 0}\delta \ f(X)(\Delta X)^{-1},$$

$$'f(X) = \lim_{\Delta X\to 0}(\Delta X)^{-1}\Delta \ f(X),$$

where the limits are taken with $|\Delta X| \to 0$ along the direction of the four-vector ΔX which depends on the 3-dimensional vector $\Delta\vec{X}$. Following the same complex procedure, take the derivatives along a fixed direction $\Delta X = \Delta X_\beta\, e^\beta$ (no sum on β), indicated by the index within parenthesis:

$$f'(X)_{(\beta)} = \frac{\partial U_0}{\partial X_\beta}\, e^0(e^\beta)^{-1} + \sum_i \frac{\partial U_i}{\partial X_\beta}\, e^i(e^\beta)^{-1},$$

$$'f(X)_{(\beta)} = \frac{\partial U_0}{\partial X_\beta}(e^\beta)^{-1}\, e^0 + \sum_i \frac{\partial U_i}{\partial X_\beta}(e^\beta)^{-1}\, e^i.$$

Straightforward calculation shows that

$$f'(X)_{(0)} = {}'f(X)_{(0)},$$

$$f'(X)_{(j)} = {}'f(X)_{(j)} - 2\sum_{i,k}\epsilon^{ijk}\frac{\partial U_i}{\partial X_j}\, e^k$$

By imposing that these derivatives are selectively equal, four basic classes of complex-like analytic functions can be obtained:

Right analytic	Left analytic	Left–right analytic	Total analytic
$f'(X)_{(0)} = f'(X)_{(i)}$ $f'(X)_{(i)} = f'(X)_{(j)}$	$'f(X)_{(0)} = {}'f(X)_{(i)}$ $'f(X)_{(i)} = {}'f(X)_{(j)}$	$f'(X)_{(\alpha)} = {}'f(X)_{\beta}$	$f'(X)_{(\alpha)} = f'(X)_{(\beta)}$, $'f(X)_{(\alpha)} = {}'f(X)_{(\alpha)}$ $'f(X)_{(\alpha)} = f'(X)_{(\alpha)}$
$\frac{\partial U_\alpha}{\partial X_\alpha} = \frac{\partial U_\beta}{\partial X_\beta}$	$\frac{\partial U_\alpha}{\partial X_\alpha} = \frac{\partial U_\beta}{\partial X_\beta}$	$\frac{\partial U_\alpha}{\partial X_\alpha} = \frac{\partial U_\beta}{\partial X_\beta}$	$\frac{\partial U_\alpha}{\partial X_\alpha} = \frac{\partial U_\beta}{\partial X_\beta}$
$\frac{\partial U_i}{\partial X_0} = -\frac{\partial U_0}{\partial X_i}$	$\frac{\partial U_i}{\partial X_0} = -\frac{\partial U_0}{\partial X_i}$	$\frac{\partial U_i}{\partial X_j} = -\frac{\partial U_j}{\partial X_i}$	$\frac{\partial U_\alpha}{\partial X_\beta} = 0 \quad \alpha \neq \beta$
$\frac{\partial U_i}{\partial X_j} = \sum \epsilon^{ijk} \frac{\partial U_k}{\partial X_0}$	$\frac{\partial U_i}{\partial X_j} = -\sum \epsilon^{ijk} \frac{\partial U_k}{\partial X_0}$	$\frac{\partial U_i}{\partial X_0} = -\frac{\partial U_0}{\partial X_i}$	

As we have mentioned these conditions are too restrictive for most applications, including quantum mechanics.

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